Exercise 22

Solve the differential equation using (a) undetermined coefficients and (b) variation of parameters.

$$y'' - y' = e^x$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y'' - y'_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y'_c = re^{rx} \quad \rightarrow \quad y''_c = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} - re^{rx} = 0$$

Divide both sides by e^{rx}.

$$r^2 - r = 0$$

Solve for r.

$$r(r-1) = 0$$

$$r = \{0, 1\}$$

Two solutions to the ODE are $e^0 = 1$ and e^x. By the principle of superposition, then,

$$y_c(x) = C_1 + C_2 e^x.$$

On the other hand, the particular solution satisfies the original ODE.

$$y''_p - y'_p = e^x \tag{2}$$

Part (a)

Since the inhomogeneous term is an exponential function, the particular solution would be $y_p = Ae^x$. e^x is part of the complementary solution, though, so a factor of x is needed: $y_p = Axe^x$.

$$y_p = Axe^x \quad \rightarrow \quad y'_p = A(x+1)e^x \quad \rightarrow \quad y''_p = A(x+2)e^x$$

Substitute these formulas into equation (2).

$$[A(x+2)e^x] - [A(x+1)e^x] = e^x$$

$$Ae^x = e^x$$

Match the coefficients on both sides to get an equation for A.

$$A = 1$$

The particular solution is

$$y_p = xe^x.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1 + C_2 e^x + xe^x, \end{aligned}$$

where C_1 and C_2 are arbitrary constants.

Part (b)

In order to obtain a particular solution, use the method of variation of parameters: Allow the parameters in the complementary solution to vary.

$$y_p = C_1(x) + C_2(x)e^x$$

Differentiate it with respect to x.

$$y_p' = C_1'(x) + C_2'(x)e^x + C_2(x)e^x$$

If we set

$$C_1'(x) + C_2'(x)e^x = 0, \tag{3}$$

then

$$y_p' = C_2(x)e^x.$$

Differentiate it with respect to x once more.

$$y_p'' = C_2'(x)e^x + C_2(x)e^x$$

Substitute these formulas into equation (2).

$$[C_2'(x)e^x + \cancel{C_2(x)e^x}] - [\cancel{C_2(x)e^x}] = e^x$$

Simplify the result.

$$C_2'(x)e^x = e^x \tag{4}$$

Solve for $C_2'(x)$.

$$C_2'(x) = 1$$

Integrate this result to get $C_2(x)$, setting the integration constant to zero.

$$C_2(x) = x$$

Solve equation (3) for $C_1'(x)$.

$$\begin{aligned} C_1'(x) &= -C_2'(x)e^x \\ &= -(1)e^x \end{aligned}$$

Integrate this result to get $C_1(x)$, setting the integration constant to zero.

$$C_1(x) = -e^x$$

Therefore,

$$\begin{aligned} y_p &= C_1(x) + C_2(x)e^x \\ &= (-e^x) + (x)e^x \\ &= e^x(x-1), \end{aligned}$$

and the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1 + C_2e^x + e^x(x-1) \\ &= C_1 + (C_2 - 1)e^x + xe^x \\ &= C_1 + C_3e^x + xe^x, \end{aligned}$$

where C_1 and C_3 are arbitrary constants.